Coordinated Particle Relocation Using Finite Static Friction with Boundary Walls*

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Abstract

We present methods for achieving *arbitrary* reconfiguration of two particles in convex workspaces, based on the use of external forces, such as a magnetic field or gravity. This concept can be used for a wide range of applications in which particles do not have their own energy supply.

A crucial challenge for achieving any desired target configuration is breaking global symmetry in a controlled fashion. Previous work made use of specifically placed barriers; however, introducing precisely located obstacles into the workspace is impractical for many scenarios. In this paper, we present a different, less intrusive method: making use of the interplay between static friction with a boundary and the external force to achieve arbitrary reconfiguration. Our key contributions are a precise characterization of the critical coefficient of friction that is sufficient for rearranging two particles in triangles, convex polygons, and regular polygons.

1 Introduction

Reconfiguring a large set of objects in a prespecified manner is a fundamental task for a large spectrum of applications, including swarm robotics, smart materials and advanced manufacturing. In many of these scenarios, the involved items are not equipped with individual motors or energy supplies, so actuation must be performed from the outside. Moreover, reaching into the workspace to manipulate individual particles of an arrangement is often impractical or even impossible; instead, global external forces (such as gravity or a magnetic force) may have to be employed, targeting each object in the same, uniform manner. These limitations of individual navigation apply even in scenarios of swarm robotics: For example, the well-known kilobots do have individual actuation and energy supply, but often make use of an external light source for navigation [10]; as a consequence, directing a swarm of kilobots by switching on a light beacon works just like activating an external force. This concept of global control has also been studied for using biological cells as reactive robots controlled by magnetic fields [2, 8]. Global control also has applications in assembling nano- and micro-structures. Related work shows how to assemble shapes by adding one particle at a time [7, 4], or combining multiple pairs of subassemblies in parallel in one time step [12].

Considering this approach of navigation by a global external force gives rise to a number of problems, including navigation of one particle from a start to a goal position [9], particle computation [5, 6], or emptying a polygon [1]. Zhang et al. [15, 16] show how to rearrange a rectangle of agents in a workspace that is only constant times larger than the number of agents.

* A video showing context and animations of our results can be found in [3].

† Work from these authors was partially supported by National Science Foundation IIS-1553063 and IIS-1619278.

36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020.
This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

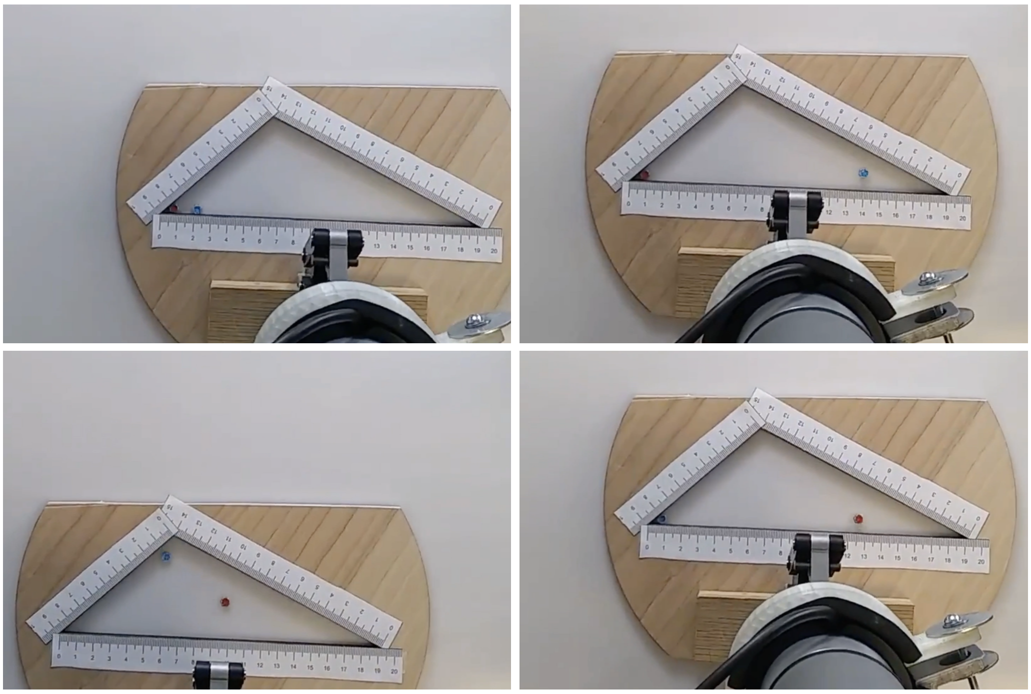


Figure 1 A robot arm moving a triangle to reconfigure two particles. Top left: Two particles are close together. Top right: Blue particles has been separated from the red particle with zig-zag moves. Bottom left: Situation after a south-east and a south-west move. Bottom right: After the blue particle is kept in the bottom left corner, the red particles is moved away with zig-zag moves.

A crucial issue for all these tasks is how to combine the use of a uniform force (which is the same for all involved items) with the individual requirements of object relocation (which may be distinct for different particles): How can we achieve an *arbitrary* arrangement of particles if all of them are subjected to the same external force? Previous work (such as [6]) has shown how arbitrary reconfiguration of an ensemble is possible with the help of specifically placed barriers; however, introducing precisely located obstacles into the workspace is impractical for many scenarios. In this paper, we present a different, less intrusive method: making use of the interplay between static friction with a boundary of the workspace and the external force to achieve any desired configuration. A real-world example is shown in Figure 1.

Shahrokhi et al. [13, 14] already considered reconfiguration problems of particles using friction at the walls. However, they assume walls have infinite friction, i.e., a particle lying at a wall cannot be moved when there is a movement parallel to the wall. This differs from the more realistic assumptions in this paper, in which we only consider finite friction as in [11]

1.1 Our Results.

We provide a fundamentally new approach to manipulating a swarm of objects by an external, global force, demonstrating how static boundary friction can be employed to achieve arbitrary reconfiguration. Our results include the following.

- We show that any two particles in an arrangement can be arbitrarily relocated in a convex workspace, provided sufficient friction as a function of the geometry.
- More specifically, for a triangle with second smallest angle β, we prove that an angle of friction of $\frac{\pi}{2} - \beta$ is always sufficient to guarantee any reconfiguration.

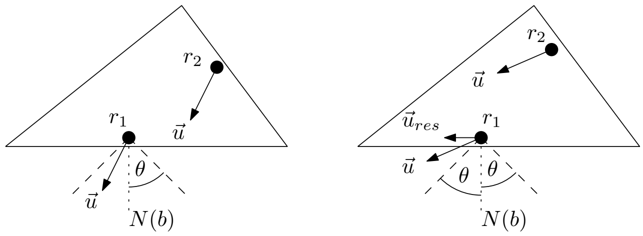


Figure 2 Left: An input force command $u(t)$ within the cone $\pm\theta$ about the normal to the boundary results in no motion of r_1. Right: An input force command $u(t)$ outside the cone results in a motion of both particles. Observe that r_1 slides along the boundary with a resulting force $u_{\text{res}}(t)$.

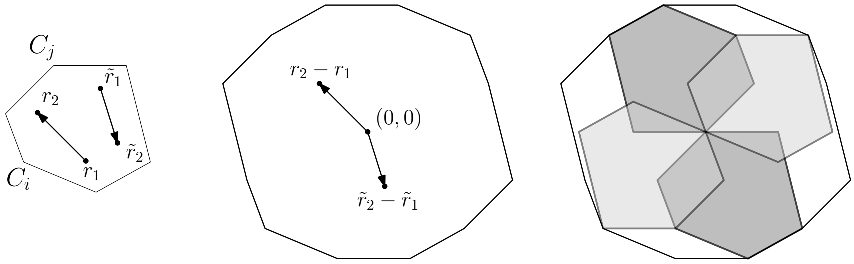


Figure 3 Left: A six-sided polygon P with start positions r_1 and r_2 for two particles and their goal positions $\tilde{r}_1$ and $\tilde{r}_2$. Middle: The Δ configuration of the polygon and the positions of the start and end configuration. Right: Lightgray (darkgray) area corresponds to the C_i-area (C_j-area, resp.).

2 Preliminaries

▶ **Definition 1.** Let θ be the *angle of friction* and $\mu := \tan\theta$ be the *coefficient of friction*. For a particle r lying at a boundary side b, let N(b) be the normal to b. If the angle between force command $\vec{\mathrm{u}}$ and N(b) is at most θ, then r does not move at all. If the angle is larger than $\frac{\pi}{2}$ then r moves with full speed. In this paper we do not consider the remaining case.

▶ Problem 1. Given a *workspace*, i.e., a convex polygon with n corners $C_1, \ldots, C_n$, particles r_1 and r_2, and an angle of friction θ, is it possible to reach the configuration $\tilde{r}_1$ and $\tilde{r}_2$?

In this paper, we do not make any assumption on the initial positions of r_1 and r_2, except that they are well separated, i.e., they have a distance $\varepsilon > 0$ to each other.

▶ **Definition 2** (Δ Configuration)**.** The Δ configuration space Δ_P of a convex polygon P with vertices $C_1, \ldots, C_n$ is defined as $\Delta_P := \mathrm{ch}\,(C_i - C_j \mid C_i, C_j \in P)$, where $\mathrm{ch}(\cdot)$ denotes the convex hull (for an example see Figure 3). This gives us the set of all relative positions of r_2 to r_1.

From this definition follows that $\Delta_P = \Delta_{-P}$, where $-P$ is P rotated by π. This motivates the following definition.

▶ **Definition 3.** Let C be some vertex of P. The C-area in Δ_P is the union of P and $-P$ having C centered at the origin (see Figure 3 right).

Note that the union of C-areas for all $C \in P$ equals Δ_P.

3 Reconfiguration of two particles

Just like in the context of sorting algorithms in computer science or discrete mathematics, a critical component for achieving arbitrary reconfiguration of larger ensembles is the ability to rearrange two specific particles. For our purposes of employing external forces and static friction, the additional aspects of geometry and physics have to be considered. These are addressed in this section.

The main idea for this first step is to try to completely cover the Δ configuration. We start by developing a strategy for separating two particles in Subsection 3.1, which gives us a lower bound for θ for every strategy in this section. This is followed by an upper bound for θ in triangles (Subsection 3.1) and arbitrary convex polygons (Subsection 3.2), i.e., we can guarantee any reconfiguration with any angle of friction higher than this bound. By each strategy we develop, more parts of the Δ configuration are covered. Thus, our goal is to give strategies, whose union of covered areas is exactly the Δ configuration.

3.1 Reconfiguration of two particles in arbitrary triangles

As a first step, we provide a sufficient large angle of friction to separate two specific particles.

▶ **Lemma 4.** *Assume particle r_1 is positioned in a corner with angle α, then we can move r_2 to any position in the polygon without moving r_1 by performing zig-zag moves, if $\theta > \frac{\alpha}{2}$ (see Fig. 1 and 4a).*

Proof. Omitted due to space constraints. ◀

Now, let T be a triangle with corners A, B and C, and angles α, β and γ. Furthermore, let α be the smallest angle in T and we assume that $\theta > \frac{\alpha}{2}$ is guaranteed. Consider two particles r_1 and r_2 within T and their goal positions $\tilde{r}_1$ and $\tilde{r}_2$. We have the following strategies to reach the goal positions (see also Fig. 4 for a graphical sketch):

Blue: Move r_1 to A. As shown in Figure 4a, use zig-zag moves to place r_2 in T while r_1 is fixed in A, such that $r_2 - r_1 = \tilde{r}_2 - \tilde{r}_1$. Then, translate r_1 and r_2 to their goal positions.

Red: First, place r_2 in A and move r_1 to B. Then, place r_2 anywhere in the area spanned by $\overline{AB}$ and the angle $\frac{\pi}{2} - \beta + \theta$. Afterwards, translate r_1 and r_2 to their goal positions.

Green: First, Place r_2 in A and move r_1 to C. Then, place r_2 in the area spanned by $\overline{AC}$ and the angle $\frac{\pi}{2} - \gamma + \theta$, such that $r_2 - r_1 = \tilde{r}_2 - \tilde{r}_1$. Afterwards, translate r_1 and r_2 to their goal positions.

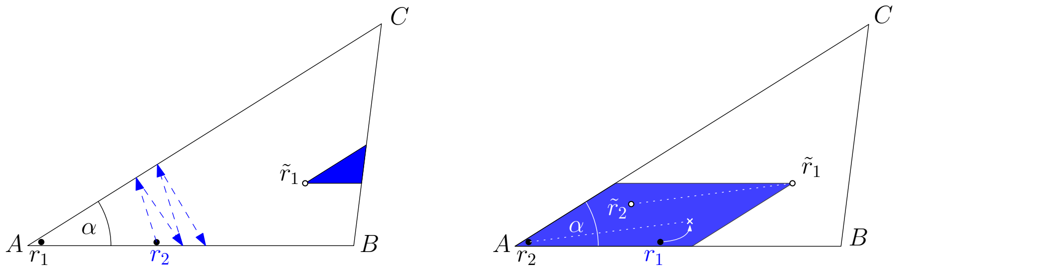
Orange: Place r_2 in C and r_1 in B (as we will see later, this is always possible if $\theta > \frac{\alpha}{2}$). Then, place r_2 in the area spanned by $\overline{BC}$ and the angle $\frac{\pi}{2} - \beta + \theta$, such that $r_2 - r_1 = \tilde{r}_2 - \tilde{r}_1$. Afterwards, translate both particles to their goal position.

Violet: Place r_2 in B and r_1 in C. Then, place r_2 anywhere in the area spanned by $\overline{CB}$ and the angle $\frac{\pi}{2} - \gamma + \theta$, such that $r_2 - r_1 = \tilde{r}_2 - \tilde{r}_1$ Finally, translate both particles to their goal position.

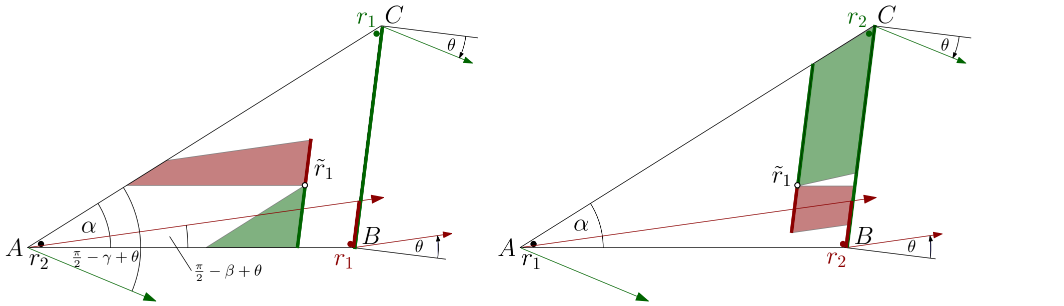
These strategies can also be used by switching the particles r_1 and r_2: Assume that r_1 lies in corner A. To switch r_1 and r_2, we separate both particles to corners B and C, then we use strategy orange or violet (depending on which particle is in which corner), and as a last step, we move r_2 to A.

▶ **Observation 1.** *In the Δ configuration, the covered areas of strategies that overlap are red with orange and green with violet. The blue strategy covers the A-area, red and orange cover parts of the B-area, and green and violet cover parts of the C-area (see Figure 5).*

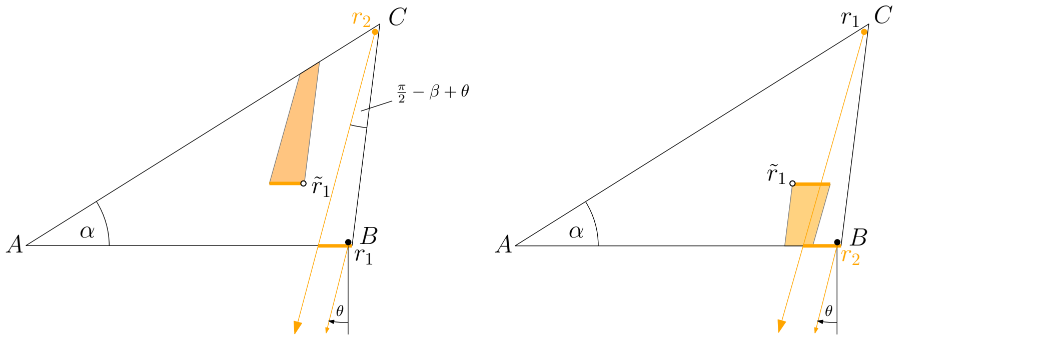
▶ **Lemma 5.** *If $\theta > \frac{\pi}{2} - \gamma$, then the area of the red and orange strategy cover the B-area.*



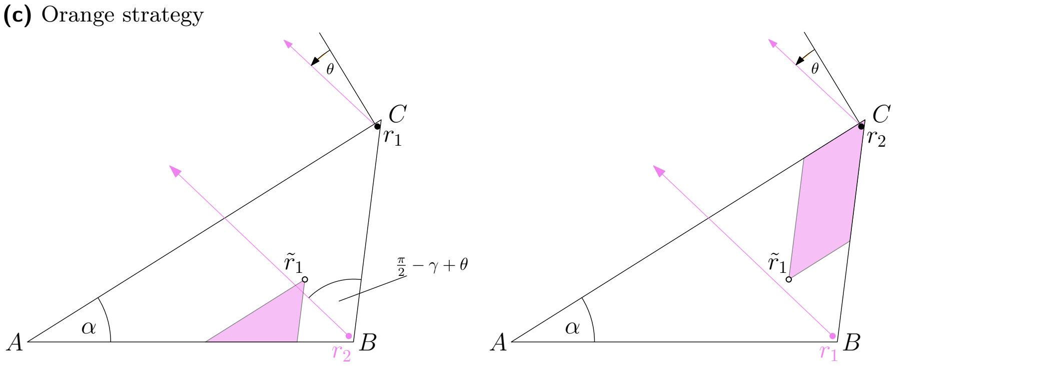
(a) Blue strategy. Dotted lines in the right figure represent the vector $\tilde{r}_2 - \tilde{r}_1$. We move r_1 to the cross with zig-zag moves and then translate both particles to their goal positions.



(b) Red and green strategy



(c) Orange strategy



(d) Violet strategy

Figure 4 Illustration of the five strategies. Colored areas correspond to valid goal positions for r_2, if the goal position of r_1 is $\tilde{r}_1$. Left column: We fix r_1 and move r_2. Right column: We switch the intermediate locations of r_1 and r_2.

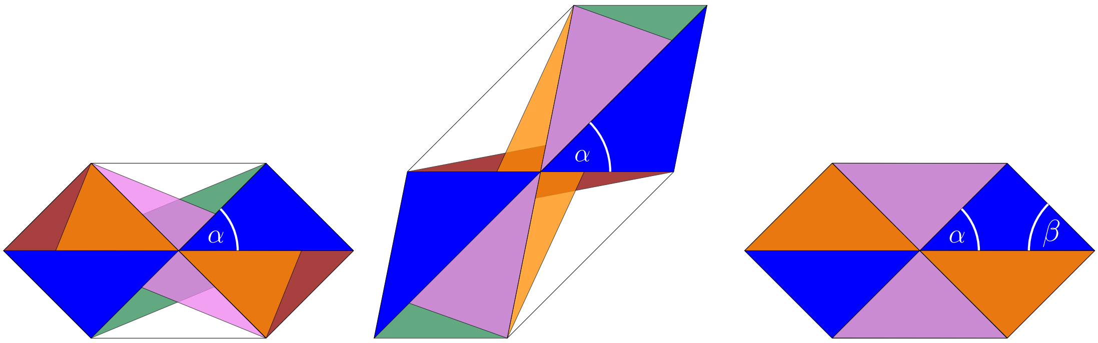


Figure 5 Shown in (a) and (b) are the Δ configurations of top right blue triangle. Colors represent the areas in the Δ configuration covered by our five strategies with an angle of friction of $\frac{\alpha}{2} + \varepsilon$ for some $\varepsilon > 0$. (a),(b): We observe that every strategy may cover areas not covered by any other strategy. (c): If $\theta > \frac{\pi}{2} - \beta$ and $\beta < \gamma$ then we can guarantee full coverage.

Proof. We can prove that the red strategy covers the B-area if $\theta > \frac{\pi}{2} - \gamma$, and that the orange strategy covers the B-area if $\theta > \frac{\pi}{2} - \alpha > \frac{\pi}{2} - \gamma$. Therefore, the lemma holds. Due to space constraints, full details are omitted. ◀

With a similar proof, we can show the following lemma:

▶ **Lemma 6.** *If $\theta > \frac{\pi}{2} - \beta$, then the area of the green and violet strategy cover the C-area.*

▶ **Theorem 7.** *Let T be a triangle with angles $\alpha \leq \beta \leq \gamma$. If $\theta > \frac{\pi}{2} - \beta$, then we can guarantee any reconfiguration of two particles, i.e., Δ_T is completely covered by our strategies.*

Proof. To cover the A-, B-, and C-area of the Δ configuration, the angle of friction θ must be greater than $\max(\frac{\alpha}{2}, \frac{\pi}{2} - \beta, \frac{\pi}{2} - \gamma)$. This is true for $\theta > \frac{\pi}{2} - \beta$. ◀

Because $\frac{\pi}{2} - \gamma = \frac{\pi - 2\gamma}{2} \leq \frac{\pi - \gamma - \beta}{2} = \frac{\alpha}{2}$, the B-area is always covered if $\theta > \frac{\alpha}{2}$. This leads to the following corollary.

▶ **Corollary 8.** *For a triangle T with angles $\alpha \leq \beta \leq \gamma$, at least two thirds of all configurations can be guaranteed if $\theta > \frac{\alpha}{2}$.*

3.2 Reconfiguration of two particles in convex polygons

In this section we generalize the strategy for triangles, i.e., for a particle r_1 in corner C_i and a particle r_2 in corner C_j, moving particle r_2 to cover the C_i-area. As shown in Figure 6, we cannot guarantee full coverage with this strategy, because any movement for r_2 in direction to C_1 would also move r_1. This happens for all pairs of vertices (C_i, C_j) of P, where the segment C_jC_{j+1} has a larger negative slope than the segment C_iC_{i-1}, i.e., if the sum of exterior angles between vertices C_i and C_j is smaller than γ_i. This motivates the following definition.

▶ **Definition 9.** For a vertex $C_i \in P$, let δ_i be the exterior angle at vertex C_i. Let $P^+_{i,j} := \{C_i, C_{i+1}, \ldots, C_{j-1}, C_j\}$ and $P^-_{i,j} := \{C_i, C_{i-1}, \ldots, C_{j+1}, C_j\}$. We define

$$\eta^+_{i,j} := \sum_{C_k \in P^+_{i+1,j-1}} \delta_k \text{ and } \eta^-_{i,j} := \sum_{C_k \in P^-_{i-1,j+1}} \delta_k$$

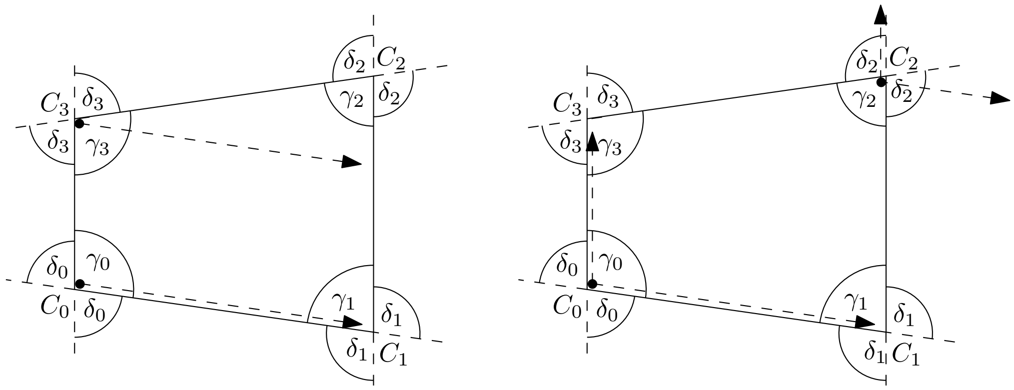


Figure 6 Left: Particle in C_0 cannot be moved without moving a particle in C_3, because $\delta_0 < \gamma_3$. Right: However, we can move the particle in C_0 to any place in the polygon without moving a particle in C_2 (unless the friction is too small), because $\delta_3 + \delta_0 > \gamma_2$ and $\delta_1 + \delta_0 > \gamma_2$.

Furthermore, let $P_i := \{C_j \in P \mid \eta^+_{i,j+1} \geq \gamma_i \wedge \eta^-_{i,j-1} \geq \gamma_i\}$, i.e., P_i contains every vertex of P such that we can use the strategy described in the beginning of this section. Note that all indices are modulo n.

▶ **Lemma 10.** *For a vertex C_i of P, we have $|P_i| \geq 1$.*

Proof. Assume that $|P_i| = 0$. W.l.o.g., let j be the largest index, such that $\eta^+_{i,j+1} < \gamma_i$. If $\eta^-_{i,j} > \gamma_i$, then it immediately follows that $C_{j+1} \in P_i$ and $|P_i| \geq 1$. Otherwise, we have two adjacent vertices C_j and C_{j+1} such that $\eta^+_{i,j+1} < \gamma_i$ and $\eta^-_{i,j} < \gamma_i$. This implies that $2\gamma_i > \eta^+_{i,j+1} + \eta^-_{i,j-1} = -\delta_i + \sum\limits_{C_k \in P} \delta_k = -\delta_i + 2\pi > 2\pi - 2\delta_i = 2\gamma_i$. This is a contradiction and therefore $|P_i| \geq 1$. ◀

▶ **Lemma 11.** *Let P be a convex polygon with vertices $C_0, \dots, C_{n-1}$ and angles $\gamma_0, \dots, \gamma_{n-1}$. We can cover the C_i-area if $\theta > \min\limits_{j \in P_i} \left(\frac{\gamma_i}{2}, \max\left(\frac{\gamma_j}{2}, \eta^+_{i,j} - \frac{\pi}{2}, \eta^-_{i,j} - \frac{\pi}{2}\right)\right)$.*

Proof. Omitted due to space constraints. ◀

Combining Lemmas 10 and 11 yields the following theorem.

▶ **Theorem 12.** *Let P be a convex Polygon with vertices $C_0, \dots, C_{n-1}$ and angles $\gamma_0, \dots, \gamma_{n-1}$. If $\theta > \max\limits_{0 \leq i < n} \left(\min\limits_{j \in P_i} \left(\frac{\gamma_i}{2}, \max\left(\frac{\gamma_j}{2}, \eta^+_{i,j} - \frac{\pi}{2}, \eta^-_{i,j} - \frac{\pi}{2}\right)\right)\right)$, then every configuration of two particles can be reached.*

▶ **Corollary 13.** *If P is a regular polygon with n vertices and if $\mu > \cot(\pi/n)$, then every reconfiguration is possible.*

4 Conclusion

We introduced a novel approach for rearranging the positions of particles by applying global uniform forces, making use of different local static friction to achieve arbitrary goal positions. To this end, we provided strategies enabling arbitrary rearrangements of two particles in convex workspaces, giving a characterization of the critical coefficient of friction in terms of the boundary geometry. Future work can now investigate optimal motion planning.

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